Review of Univariate Calculus
Some Point Set Preliminaries

Mathematical Workshop for Graduate Students in Statistics

Tigran Markaryan

Department of Statistics
George Mason University

Document Version 5.0

September, 2018

Contents

Chapter 1

Univariate Calculus

This chapter is primarily devoted to Calculus of functions of a single variable. In several locations, however, we will point out how certain concepts or techniques generalize to higher dimensions thus planting the seed for the chapter on multivariate Calculus.

While it may be a slight oversimplification, Calculus is concerned with two types of problems: finding a tangent to a curve (or a higher dimensional surface) and finding an area under a curve (or a surface). The former makes up the Differential Calculus; the latter — the Integral Calculus. This chapter will cover the key elements of both.

There are eight sections in this Chapter. Section **??** introduces limits and continuity and their basic properties. Before these two concepts are introduced, however, necessary foundation of the topology of the real line (open and closed sets; bounded and unbounded sets; inverse and composite functions) is provided in the beginning of the section. Differentiation of functions is the topic of Section **??**. Here, properties of differentiable functions and differentiation rules (such as the chain rule) are presented. In this version, a new section on the Newton-Raphson method is added. Section **??** is dedicated to indefinite and definite integral. The properties of the integrals and several methods of evaluation (the fundamental theorem of calculus, change of variable, integration by parts) are included in the section. Improper integrals are also included. The short Section **??** introduces infinite series and important convergence results. A large number of exercises is included in Section **??**.

1.1 Some Point Set Preliminaries

In this section, we start with basic definitions of point sets on the real line and in higher dimensions. The following elements are introduced and discussed: limit points, interior and exterior, boundary, open and closed sets, bounded and unbounded sets, inverse and composite functions.

1.1.1 Distance in One- and Higher-Dimensional Spaces

We define the distance between two real numbers a and b as $|a-b|$ and we define the magnitude of a real number a as its distance from 0, i.e., $|a|$. The distance satisfies the triangle inequality, namely

$$|a-b| \leq |a-c| + |c-b| \qquad \forall c \in \mathbb{R} \tag{1.1}$$

These concepts generalize to higher-dimensional spaces in a natural fashion and we provide those dimensions in the balance of this section.

The most common distance (but not the only one[1]) in $\mathbb{R}^2$ is the Euclidean distance. The Euclidean distance represents the length of the straight line joining two points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ and by the Pythagorean theorem is given by:

$$d(a,b) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2}.$$

This definition extends naturally to higher dimensions.

Definition 1.1 (Euclidean distance). *The Euclidean distance between $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ is*

$$d(a,b) = \sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + \cdots + (a_n-b_n)^2} = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

The triangle inequality holds for any $a, b, c \in \mathbb{R}^n$ (The proof is left as an exercise):

$$d(a,b) \leq d(a,c) + d(c,b) \tag{1.2}$$

[1]A distance in $\mathbb{R}^n$ can be defined by any nonnegative function $\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that possesses the following three defining properties: 1) $\rho(a,b) = 0 \iff a = b$ (identity), 2)$\rho(a,b) = \rho(b,a)$ (symmetry) and 3) $\rho(a,b) \leq \rho(a,c) + \rho(c,b)$ (triangle inequality). All these properties are natural and reasonable to require for a distance function to possess. For example, the following function defines a distance that is sometimes known as a Manhattan distance (can you guess why?): $\rho(a,b) = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|$. The proof of this fact is left as an exercise.

Next we define the length or magnitude of a vector. In the case of $\mathbb{R}$ we define the magnitude (or the absolute value) of a real number as its distance from the origin. This exact same definition extends to higher dimensions. The term Euclidean norm is also used frequently when referring to the length of a vector. The qualifier 'Euclidean' is not redundant as the length (or norm) is defined using the Euclidean distance. As the Euclidean distance is the only distance we will be considering, there is no danger of confusion to omit the qualifier 'Euclidean' and that is exactly what we will do.

DEFINITION 1.2 (Length of a vector). *If $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$, then we define its length, denoted by $|a|$ or $\|a\|$, as its distance from the origin:*

$$\|a\| = d(a, 0) = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

When $n = 1$, we get the usual definition of the absolute value of a real number (verify it). The distance between two vectors can be expressed as the length of the vector that is the difference of the original two vectors:

$$d(a, b) = \|a - b\|.$$

1.1.2 Interior, Exterior, Limit Points and Boundary

As a quick refresher, in this subsection, we introduce the concepts of a neighborhood, limit points, boundary points and the interior and exterior of a set.

DEFINITION 1.3 (Neighborhood of a point). *If $x \in \mathbb{R}$, then for any $h > 0$ the open interval $(x - h, x + h)$ is called a neighborhood of x.*

In higher dimensions, the interval is replaced with a ball. So, if $x \in \mathbb{R}^n$, then for any $h > 0$, the open ball $\{y \in \mathbb{R}^n : \|y - x\| < h\}$ is called a neighborhood of x. When there is no room for confusion, we may also use $||$ in place of $\|\|$ to denote the length of a vector.

DEFINITION 1.4 (Interior point). *Let $A \subset \mathbb{R}$ and let $x \in A$. Then x is called an interior point of A, if there exists a neighborhood of x that is entirely contained in A.*

QUESTION 1.1. *Can a set have only one interior point? Can it have only finite number of interior points?*

DEFINITION 1.5 (Exterior point). *Let $A \subset \mathbb{R}$ and let $x \in \mathbb{R}$. Then x is called an exterior point of A, if there exists a neighborhood of x that does not contain any points from A.*

QUESTION 1.2. *Can an exterior point of a set belong to the set?*

QUESTION 1.3. *Can a set have only one exterior point? Can it have only finite number of exterior points?*

DEFINITION 1.6 (Boundary point). *Let $A \subset \mathbb{R}$ and let $x \in \mathbb{R}$. Then x is called a boundary point of A, if any neighborhood of x contains points from both A and A^c.*

QUESTION 1.4. *Can a set have only one boundary point?*

It should be clear that any set A induces a partition of $\mathbb{R}$ into the following three entities: 1) interior of A, 2) exterior of A and 3) boundary of A. This is left as an exercise.

DEFINITION 1.7 (Limit point of a set). *Let A be a set in $\mathbb{R}$ and let $x \in \mathbb{R}$. Then x is called a limit point of A, if any neighborhood of x contains at least one point from A that is distinct from x.*

QUESTION 1.5. *Does a boundary point have to be a limit point? How about the converse?*

The definitions of interior, exterior, boundary and limit points extend verbatim to higher dimensions.

Note that in the definition of the limit point, it is not required for $x \in A$. Limit points are also sometimes called *accumulation points*. From the definition it should be clear that there are infinitely many points from set A in any neighborhood of any limit point of A (the proof is left as an exercise). Next we look at some examples and identify the interior, exterior, boundary and limit points of the given sets.

EXAMPLE 1.1 (Interior, exterior, boundary and limit points of sets). *Find the interior, exterior, boundary and limit points of the following sets.*

i) *The closed interval* $[a, b]$

ii) *The open interval* (a, b)

iii) *The set of the natural numbers* $\mathbb{N}$.

iv) *The rational numbers between 0 and 1.*

v) *The set* $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$.

vi) The set $\left\{n \in \mathbb{N} : \left(1 + \frac{1}{n}\right)^n\right\}$.

vii) The set $(0,1) \cup (1,2) \cup (2,3)$.

viii) The set consisting of a single point: $\{1\}$.

Solution. PROVIDE □

1.1.3 Open and Closed Sets on Real Line

We are now ready to define open and closed sets in $\mathbb{R}$.

Definition 1.8 (Open sets in $\mathbb{R}$). *Let A be a set in $\mathbb{R}$. If all points of A are interior points, then A is called an open set.*

Definition 1.9 (Closed sets in $\mathbb{R}$). *Let A be a set in $\mathbb{R}$. If A contains all its limit points, then it is called a closed set.*

We now revisit Example 1.1 and pose additional questions on whether those sets are open, closed or neither.

Example 1.2 (Open and closed sets). *Investigate whether the following sets are open or closed.*

i) The closed interval $[a,b]$

ii) The open interval (a,b)

iii) The set of the natural numbers $\mathbb{N}$.

iv) The rational numbers between 0 and 1.

v) The set $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$.

vi) The set $\left\{n \in \mathbb{N} : \left(1 + \frac{1}{n}\right)^n\right\}$.

vii) The set $(0,1) \cup (1,2) \cup (2,3)$.

viii) The set consisting of a single point: $\{1\}$.

Solution. PROVIDE □

There is a simple relationship between open and closed sets: a set is open if and only if its complement is closed (the proof is left as an exercise). Some additional and useful properties of open and closed sets are given in the next proposition. The proof is left as an exercise. You should be able to observe duality between the open and closed sets and the operations of union and intersection.

PROPOSITION 1.1 (Properties of open and closed sets).

i) The union of any collection (finite, countably infinite or uncountably infinite) of open sets is open.

ii) The intersection of a finite collection of open sets is open. (This property is not true when infinite collection is considered.)

iii) The union of a finite collection of closed sets is closed. (This property is not true when infinite collection is considered.)

iv) The intersection of any collection (finite, countably infinite or uncountably infinite) of closed sets is closed.

Proof. Left as an exercise. □

It is not difficult to come up with examples that show that an infinite intersection of open sets does not have to be an open set or an infinite union of closed sets does not have to a closed set. Let

$$A_n = \{(-1/n, 1/n)\} \quad n \in \mathbb{N}.$$

Then it is easy to check that (show it!)

$$\bigcap_{n=1}^{\infty} A_n = \{0\}.$$

A singleton is not an open set. Thus, an infinite intersection of open sets does not have to be an open set. Similarly, an infinite union of closed sets does not have to be a closed set as the following example demonstrates.

$$B_n = \{[0, 2 - 1/n]\} \quad n \in \mathbb{N}.$$

$$\bigcup_{n=1}^{\infty} B_n = \{[0, 2)\}.$$

The semiopen interval $\{[0,2)\}$ is not closed.

QUESTION 1.6. *Can a nested intersection of nonempty open sets be the empty set?*

QUESTION 1.7. *Can a set be both closed and open?*

1.1.4 Bounded and Unbounded Sets on Real Line

A set $A \subset \mathbb{R}$ is said to be *bounded* if there exists a number $M > 0$, such that

$$\forall x \in A \Rightarrow |x| \leq M.$$

In other words, a set is bounded if it is contained in some interval centered at 0. This definition extends to $\mathbb{R}^n$ without any changes as long as one interprets $|x|$ as the Euclidian length of vector x. Sets that are not bounded are called *unbounded*. Next we look at several examples of bounded and unbounded sets.

EXAMPLE 1.3 (Bounded and unbounded sets). *Investigate whether the following sets are bounded or unbounded.*

i) *The closed interval* $[a,b]$

ii) *The open interval* (a,b)

iii) *The set of the natural numbers* $\mathbb{N}$.

iv) *The rational numbers between 0 and 1.*

v) *The set of positive real numbers:* $[0,\infty)$.

vi) *The set* $\left\{n \in \mathbb{N} : \left(1+\frac{1}{n}\right)^n\right\}$. *This is left as an exercise.*

vii) *The set* $(0,1) \cup (1,2) \cup (2,3)$.

viii) *The set consisting of a single point:* $\{1\}$.

ix) *The set* $\left\{n \in \mathbb{N} : \frac{1+(-2)^n}{n}\right\}$.

x) *The set of all prime numbers.*

xi) *The set of the roots of the equation:* $\sin x = \frac{1}{2}$.

<u>*Solution*</u>. PROVIDE □

We saw an example of an infinite set, namely $\mathbb{N}$ that had no limit point. But can a similar behavior occur if the set is bounded? The answer to that question is a no as the following almost 200-year old well-known result ascertains (we state it without a proof).

THEOREM 1.1 (Bolzano–Weierstrass[2]). *If a bounded set $A \subset \mathbb{R}$ has infinitely many points, then A has at least one limit point.*

Bolzano–Weierstrass theorem has many applications and also generalizes to $\mathbb{R}^n$.

1.1.5 Inverse Functions

Functions that map any two distinct points in their domain to distinct points in their range possess an important property: they have inverses. These functions are called one-to-one functions and we have already encountered them when defining cardinality of sets. Let us look at the following function with domain A and range B.

$$\begin{aligned} f : A &\to B, \\ A &= \{1, 2, 3\}, \\ B &= \{1, 8, 27\}. \end{aligned}$$

The function is defined as follows:

$$\begin{aligned} f &: 1 \to 1, \\ f &: 2 \to 8, \\ f &: 3 \to 27. \end{aligned}$$

So, f maps each of the three numbers in the domain to their cubes: $f(x) = x^3 \quad x \in A$. This function is a one-to-one function because no two points in the domain are mapped to the same point in the range. Therefore, it has an inverse function: f^{-1}. The domain of the inverse function is B and the range is A and it is given by:

$$\begin{aligned} f^{-1} &: 1 \to 1, \\ f^{-1} &: 8 \to 2, \\ f^{-1} &: 27 \to 3. \end{aligned}$$

[2]Some textbooks refer to this result as Bolzano-Weierstrass Lemma.

It is not difficult to recognize the inverse function as the cubed root function:

$$f^{-1}(x) = \sqrt[3]{x}.$$

Note that it is not material whether we use x, y or any other letter as the dummy argument of the function. It should be clear by now that the inverse function "returns" the image to its pre-image. The function f maps 2 to 8 and the function f^{-1} maps 8 back to 2. It should also be clear that applying f^{-1} to f will result in the identity mapping.

Let us look at the quadratic function: $f(x) = x^2$. Its domain is the whole $\mathbb{R}$ and its range is the set of non-negative real numbers. This function is not one-to-one, because it maps both x and $-x$ to x^2. Therefore, when considered as a function on $\mathbb{R}$, the quadratic function does not possess an inverse. If we restrict the domain of the function to either $[0, \infty)$ or $(-\infty, 0]$, then we do get one-to-one mapping and therefore inverse functions exist. For example, if $f(x) = x^2 \quad x \geq 0$, then $f^{-1}(x) = \sqrt{x}$. Of course, the quadratic function with a domain $\mathbb{R}$ and $[0, \infty)$ are actually two different functions.

There is not a universal technique for finding inverses of functions as functions can be given in a number of different ways (description of the mapping rule, explicit mapping, formula, etc.). When a function is given as a formula $y = f(x)$, sometimes one can solve and find the inverse $x = f^{-1}(y)$. Again, we mention that it is immaterial as to what letter is used to denote the argument of a function ($f(x) = x^2$, $f(a) = a^2$, $f(z) = z^2$ all denote the same function, namely the quadratic function).

Next we look at some examples of finding inverse functions.

Example 1.4 (Inverse functions). *Determine if the following functions have inverses. If they do, provide the form of the inverse function and plot it along with the original function. What can be said about the graphs of inverse functions?*

i) $f(x) = x^4 \quad x \in \mathbb{R}$

ii) $f(x) = x^4 \quad x \geq 0$

iii) $f(x) = x^5 \quad x \in \mathbb{R}$

iv) $f(x) = \dfrac{2x-1}{x+4} \quad x \neq -4$

v) $f(x) = e^x \quad x \in \mathbb{R}$

vi) $f(x) = e^x \quad x \geq 1$

vii) $f(x) = \sin x \quad x \in \mathbb{R}$

viii) $f(x) = \dfrac{1}{x} \quad x \neq 0$

ix) *The function that maps the students at GMU as of September 07, 2013 to their G numbers.*

Solution. PROVIDE □

Next we enter the realm of higher dimensions and provide two examples on finding the inverse functions. We will need this, for example, when we perform change of variables in multiple integrals. We will start with the case $f : \mathbb{R}^2 \to \mathbb{R}^2$ and consider the linear mapping:[3]

$$y = f(x),$$
$$f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \begin{pmatrix} y_1 \\ y_2 \end{pmatrix};$$

given by the following system of linear equations:

$$\begin{cases} y_1 &= 2x_1 + x_2 - 3 \\ y_2 &= x_1 - x_2 + 4 \end{cases} \tag{1.3}$$

In order to find the inverse function $f^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$f^{-1} : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};$$

we solve the system of equation (1.3) for x_1 and x_2. It is an easy exercise to find:

$$\begin{cases} x_1 &= \frac{1}{3}y_1 + \frac{1}{3}y_2 - 1/3 \\ x_2 &= \frac{1}{3}y_1 - \frac{2}{3}y_2 + 11/3 \end{cases} \tag{1.4}$$

Thus, equation (1.4) represents the inverse function we were seeking. Of course, the functions do not have to be linear, in which case one deals with a system of non-linear equations. Let us consider the following example that involves a non-linear function

[3]To be precise, this mapping is an affine mapping.

$f : \mathbb{R}^3 \to \mathbb{R}^3$:

$$y = f(x),$$
$$f : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \to \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix};$$

given by the following system of nonlinear equations:

$$\begin{cases} y_1 &= \dfrac{x_1}{x_1 + x_2 + x_3} \\ y_2 &= \dfrac{x_2}{x_1 + x_2 + x_3} \\ y_3 &= x_1 + x_2 + x_3 \end{cases} \tag{1.5}$$

The domain of this function is

$$A = \{(x_1, x_2, x_3) : x_1 > 0, x_2 > 0, x_3 > 0\}.$$

The range of the function is (show it)

$$B = \{(y_1, y_2, y_3) : y_1 > 0, y_2 > 0, y_3 > 0, y_1 + y_2 < 1\}.$$

Thus:

$$f : A \to B \text{ and therefore}$$
$$f^{-1} : B \to A.$$

In order to find f^{-1}, we need to solve the system of equations (1.5) for (x_1, x_2, x_3). The result is:

$$\begin{cases} x_1 = y_1 y_3 \\ x_2 = y_2 y_3 \\ x_3 = y_3(1 - y_1 - y_2) \end{cases} \tag{1.6}$$

It is a useful exercise to solve the system (1.5) and confirm that the solution is indeed (1.6). We will return to this example when calculating triple integrals later.

1.1.6 Composite Functions

Let us look at following three sets.

A : The set of all art museums in the US.
B : The set of all current US Postal ZIP codes.
C : The set of all US States and Washington DC.

Let us also consider the following two functions.

$f : A \to B$ f maps each museum to the ZIP code of its address.
$g : B \to C$ g maps each ZIP code to the US State in which it is located.

For example, f maps the *National Museum of Natural History* to ZIP code 20560. In turn, g maps ZIP code 20560 to Washington, DC. Note that the range of f is included in the domain of g, therefore it is meaningful to define the function:

$$F(x) = g(f(x)$$

that maps A into C. For example, F maps the National Museum of Natural History to Washington, DC. The function $F : A \to C$ is called the composite function. The composite function is sometimes denoted as

$$g \circ f(x).$$

We will be using the notation $g(f(x))$. Note that in our example $f(g(x))$ is not even defined, because the range of g is not in the domain of f. This indicates that in general the two compositions: $f(g(x))$ and $g(f(x))$ are different and may or may not be defined. If f has an inverse, then it is not difficult to show that $f^{-1}(f(x)) = f(f^{-1}(x)) = I(x)$, where $I(x)$ is the identity function: $I(x) = x$. One can also look at the composition of the function with itself. For example, if $f(x) = x^3$, then $f(f(x)) = (x^3)^3 = x^9$.

Viewing or considering compositions is sometimes used to "decompose" a more complex function into less complex components. The chain rule of differentiation is a primary example of such usage.

Let us look at some examples.

EXAMPLE 1.5 (Composite functions). *In the following examples, find the composite* *function $g(f(x))$ if it exists. Also, give the domain and range of $g(f(x))$.*

i) $f(x) = x^4 \qquad g(x) = \sin x \qquad x \in \mathbb{R}$

ii) $f(x) = \sin x \qquad g(x) = x^4 \qquad x \in \mathbb{R}$

iii) $f(x) = e^x \qquad g(x) = \dfrac{1}{x} \qquad x \in \mathbb{R}, x \neq 0$

iv) $f(x) = 1 - 5x \qquad g(x) = 2x^2 - 2x \qquad x \in \mathbb{R}$

Solution. PROVIDE □

One does not have to restrict the notion of composite functions to compositions of two functions only. As long as the domains and ranges conform, one can have 3 or more levels of nesting. Going back to an earlier example $f(x) = x^3$, we can find the form of

$$f(f(f(x))) = ((x^3)^3)^3 = x^{27}.$$

Let us look at a couple of examples where the goal is to recognize the given function as a composition of perhaps simpler functions. When applying the chain rule of differentiation or performing change of variables, we frequently do just that: represent a function as a composition of two or more functions. There is not a single way to represent a function as a composite function, but frequently the choice is clear.

Example 1.6 (Recognize as composite functions). *In the following examples, represent the given function as a composite function. I.e., find functions f and g such that $F(x) = g(f(x))$.*

i) $F(x) = e^{3x^2 - x} \qquad x \in \mathbb{R}$

ii) $F(x) = 5\sin^2 x + 2\sin x - 4 \qquad x \in \mathbb{R}$

iii) $F(x) = e^{e^x} \qquad x \in \mathbb{R}$

Solution. PROVIDE □