Skills Assessment Test

Math Refresher Course for Graduate Students in Statistics
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1. Simplify the fractions:

   (a) $\dfrac{30a(x+y)^2}{105a^2(x^2-y^2)}$

   (b) $\dfrac{20a^3-45ab^2}{(2a+3b)^2}$

   (c) $\dfrac{a^4-b^4}{a^3+a^2b+ab^2+b^3}$

2. Using mathematical induction prove the following identities.

   (a) For any natural number $n \in \mathbb{N}$

   $$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   (b) For any natural number $n \geq 4$

   $$n! > 2^n.$$

   (c) For any $n \in \mathbb{N}$

   $$\int_{x=0}^{\infty} x^n e^{-x} dx = n!$$

3. Calculate the following sums.

   (a) $q+q^2+q^3+\cdots+q^n,$ where q is any real number different from 1.

   (b) $\sum_{i=1}^{\infty} q^i$, where $|q|<1$.

   (c) $1+3+5+7+\cdots+993+995+997+999.$

   (d) $\sum_{n=1}^{\infty} na^n$, where $|a|<1$.

4. Let A, B and C be three subsets of some set S.

   (a) Show that

   $$A \backslash B = A \cap B^c.$$

(b) Show that

$$A^c \cup B^c = (A \cap B)^c.$$

(c) Additionally assume that both A and B have finite number of elements. Let $|X|$ denote the number of elements of a finite set X. Show that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

(d) Show that

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

5. Find the derivatives $f'(x)$, for the following $f(x)$:

   (a) $f(x) = \sqrt{2x}\, e^{x^2}$.

   (b) $f(x) = \dfrac{\log(x)}{x} - x\tan(\pi x)$, where $\log(x)$ is the natural logarithm (base e).

   (c) $f(x) = \dfrac{1}{\left(1 + 2\sin^2 x\right)^3}$.

6. Find the values of the following definite integrals:

   (a) $\displaystyle\int_0^1 \frac{16x}{8x^2+2}\, dx.$

   (b) $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\, dx.$

   (c) $\displaystyle\int_0^1 x e^{-x/2}\, dx.$ *(Hint:* Use integration by parts)

7. Find the partial derivatives of $f(x, y) = x\log(x^2 + 2y)$:

   (a) $f'_x(x, y)$.

   (b) $f'_y(x, y)$.

   (c) $f''_{xy}(x, y)$.

8. Consider the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ that maps $F : \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix}$ and is given by

$$u = \sqrt{1 - x^2 - y^2}$$
$$v = \frac{x}{y}$$

   (a) What is the domain of F?

   (b) What is the range of F?

(c) Compute the Jacobian determinant of F:

$$J_F(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

(d) Evaluate the Jacobian determinant of F at $(x,y) = (0, \sqrt{3}/2)$.

9. Find all local extrema of the following function and classify them as either local minima or maxima.

$$f(x,y) = x^3 + y^3 - 3xy + 12$$

10. Find the global maximum (if it exists) of the following function

$$u(x,y,z,t) = xyzt;$$

where

$$\begin{aligned} x, y, z, t &\geq 0, \\ x + y + z + t &= 8 \end{aligned}$$

11. Find the values of the following double and triple integrals:

(a) $\int_0^1 \int_0^1 (x^2 + xy)\, dx\, dy.$

(b) $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$ *(Hint:* Use polar coordinate transformation.)

(c) $\iint_D 6xy\, dx\, dy.$

where D is the region in the first (positive) quadrant trapped between $y = x^2$ and $y = 2x$. *(Hint:* Sketch the region of integration, find the limits of integration and evaluate as an iterated integral.)

(d) $\iiint_K 4x\, dx\, dy\, dz.$

where K is the region in the first octant (the first octant is the set $\{(x,y,z) : x \geq 0, y \geq 0, z \geq 0\}$) that lies under the plane $2x + 3y + z = 6$.

12. Given the following matrix:

$$A = \begin{pmatrix} 2 & 6 & 1 \\ 5 & -5 & -2 \\ -2 & 3 & 1 \end{pmatrix}$$

(a) Determine if it is full rank.

(b) Find its inverse A^{-1}

(c) Solve the system of linear equations:

$$Ax = b,$$

where

$$b = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

and

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

13. Given the following matrix:

$$B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

find its eigenvalues and the determinant.

14. Every 2×2 matrix represents a mapping (a function, transformation) $\mathbb{R}^2 \to \mathbb{R}^2$. In other words it maps points in the plane to new (or same) points in the plane.

    (a) Which matrix if any doubles the lengths of the distance from the origin?

    (b) Which matrix if any shrinks the lengths of the distance from the origin in half?

    (c) Which matrix if any shifts the points to the right by one unit?

    (d) Which matrix if any rotates the points about the origin by 90 degrees counterclockwise?

    (e) Which matrix if any reflects the points about the x-axis?

    (f) Which matrix if any rotates the points about the origin by 90 degrees counterclockwise and doubles the lengths of the distance from the origin?